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Flux difference splitting for the Euler equations in generalised coordinates using a local parameterisation of the equation of state*

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Abstract. An efficient algorithm based on flux difference splitting is presented for the solution of the three-dimensional Euler equations of gas dynamics in a generalised coordinate system with a general equation of state. The scheme is based on solving linearised Riemann problems approximately and in more than one dimension incorporates operator splitting. The algorithm uses a local parameterisation of the equation of state and as a consequence requires only one function evaluation in each computational cell. The scheme has good shock capturing properties and the advantage of using body-fitted meshes. Numerical results are shown for Mach 8 flow of "equilibrium air" past a circular cylinder.

1. Introduction

In 1981 Roe [1] proposed an approximate (linearised) Riemann solver for the solution of the three-dimensional Euler equations in Cartesian coordinates with an ideal gas. A similar scheme was proposed by Glaister [2] for the Euler equations in Cartesian coordinates with general convex equations of state. A disadvantage of Glaister's scheme is that four function evaluations are required in each computational cell to approximate the first derivatives of the equation of state. For complex equations of state, e.g., curve fits for equilibrium air [3] this can prove to be an expensive overhead. We seek here to devise a scheme that requires only one function evaluation in each cell with no deterioration in the quality of the solution. This is achieved by a local parameterisation of the equation of state, in effect a "variable effective gamma" (VEG) scheme. Furthermore, the proposed scheme applies to a generalised coordinate system, and when incorporated with operator splitting leads to an efficient algorithm that has good shock-capturing properties and the advantage of using body-fitting meshes.

In Section 2 we consider the Jacobian matrix of one of the flux functions for the Euler equations in a generalised coordinate system, and in Section 3 derive an approximate Riemann solver for the solution of these equations. Finally, in Section 4 we give the numerical results achieved for a two-dimensional test problem using the scheme of Section 3.

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2. Euler equations

In this section we state the equations of motion for an inviscid, compressible fluid in three dimensions in terms of three generalised space coordinates. We also give the eigenvalues and eigenvectors of the Jacobian matrix of one of the corresponding flux functions.

2.1. Equations of flow

The three-dimensional Euler equations for the flow of an inviscid, compressible fluid can be written in generalised coordinates ξ, η, ζ as

$$(J\mathbf{w})_t + \mathbf{F}_\xi + \mathbf{G}_\eta + \mathbf{H}_\zeta = \mathbf{0}, \tag{1}$$

where

$$\mathbf{w} = (\varrho, \varrho u, \varrho v, \varrho w, e)^T, \tag{2}$$

$$\mathbf{F}(\mathbf{w}) = (\varrho U, J_\xi^x p + \varrho u U, J_\xi^y p + \varrho v U, J_\xi^z p + \varrho w U, U(e + p))^T, \tag{3}$$

$$\mathbf{G}(\mathbf{w}) = (\varrho V, J_\eta^x p + \varrho u V, J_\eta^y p + \varrho v V, J_\eta^z p + \varrho w V, V(e + p))^T, \tag{4}$$

$$\mathbf{H}(\mathbf{w}) = (\varrho W, J_\zeta^x p + \varrho u W, J_\zeta^y p + \varrho v W, J_\zeta^z p + \varrho w W, W(e + p))^T, \tag{5}$$

$$e = \varrho i + \tfrac{1}{2}\varrho(u^2 + v^2 + w^2) \tag{6}$$

and

$$U = J_\xi^x u + J_\xi^y v + J_\xi^z w, \tag{7}$$

$$V = J_\eta^x u + J_\eta^y v + J_\eta^z w, \tag{8}$$

$$W = J_\zeta^x u + J_\zeta^y v + J_\zeta^z w. \tag{9}$$

The Jacobian of the grid transformation $x = x(\xi, \eta, \zeta), y = y(\xi, \eta, \zeta), z = z(\xi, \eta, \zeta)$ from Cartesian coordinates (x, y, z) to generalised coordinates (ξ, η, ζ) is given by

$$J = \begin{vmatrix} x_\xi & y_\xi & z_\xi \\ x_\eta & y_\eta & z_\eta \\ x_\zeta & y_\zeta & z_\zeta \end{vmatrix}, \tag{10}$$

whilst the cofactors of this matrix are represented by J_ξ^x etc. The cofactor of x_ξ is given by

$$J_\xi^x = y_\eta z_\zeta - z_\eta y_\zeta, \tag{11}$$

with similar expresions for J_ξ^y etc. The quantities $(\varrho, u, v, w, p, i, e) = (\varrho, u, v, w, p, i, e)(\xi, \eta, \zeta, t)$ represent the density, velocity in the x, y and z coordinate directions, pressure,

specific internal energy and total energy, respectively, at a general position ξ, η, ζ in space and time t. In addition we have an equation of state of the form

$$p = p(\varrho, i). \tag{12}$$

2.2. *Structure of the Jacobian matrix*

We now give the Jacobian matrix of the flux function $\mathbf{F}(\mathbf{w})$, and its eigenvalues and eigenvectors, since this information, together with similar information for the Jacobian matrices of $\mathbf{G}(\mathbf{w})$ and $\mathbf{H}(\mathbf{w})$, will form the basis for the approximate Riemann solver.

The Jacobian matrix $A = \partial\mathbf{F}/\partial\mathbf{w}$ of the flux function $\mathbf{F}(\mathbf{w})$ is given by

$$A = \begin{bmatrix} 0 & J_\xi^x & J_\xi^y & J_\xi^z & 0 \\ RJ_\xi^x - uU & U + uSJ_\xi^x & uJ_\xi^y - vTJ_\xi^x & uJ_\xi^z - wTJ_\xi^x & TJ_\xi^x \\ RJ_\xi^y - vU & vJ_\xi^x - uTJ_\xi^y & U + vSJ_\xi^y & vJ_\xi^z - wTJ_\xi^y & TJ_\xi^y \\ RJ_\xi^z - wU & wJ_\xi^x - uTJ_\xi^z & wJ_\xi^y - vTJ_\xi^z & U + wSJ_\xi^z & TH_\xi^z \\ U(R - H) & HJ_\xi^x - uUT & HJ_\xi^y - vUT & HJ_\xi^z - wuT & U + UT \end{bmatrix} \tag{13}$$

where for convenience we have written

$$R = a^2 - \frac{p_i}{\varrho}(H - q^2), \tag{14}$$

$$S = 1 - \frac{p_i}{\varrho}, \tag{15}$$

$$T = \frac{p_i}{\varrho}. \tag{16}$$

The fluid speed q, enthalpy H and sound speed a are given by

$$q^2 = u^2 + v^2 + w^2, \tag{17}$$

$$H = \frac{p}{\varrho} + i + \tfrac{1}{2}q^2, \tag{18}$$

and

$$a^2 = \frac{pp_i}{\varrho^2} + p_\varrho. \tag{19}$$

and the quantities p_i, p_ϱ denote the derivatives $\partial p(\varrho, i)/\partial i|_p$, $\partial p(\varrho, i)/\partial p|_i$, respectively. The eigenvalues of A are given by

$$\lambda_{1,2,3,4,5} = U \pm aD, U, U, U, \tag{20a–e}$$

with corresponding linearly independent eigenvectors

$$\mathbf{e}_{1,2} = \left(1, u \pm \frac{aJ_\xi^x}{D}, v \pm \frac{aJ_\xi^y}{D}, w \pm \frac{aJ_\xi^z}{D}, H \pm \frac{aU}{D}\right)^T, \tag{21a–b}$$

$$\mathbf{e}_3 = \left(1, u, v, w, \tfrac{1}{2}q^2 + i - \frac{\varrho p_\varrho}{p_i}\right)^T, \tag{21c}$$

$$\mathbf{e}_4 = (0, -J_\xi^y, J_\xi^x, 0, vJ_\xi^x - uJ_\xi^y)^T \tag{21d}$$

and

$$\mathbf{e}_5 = (0, -J_\xi^z, 0, J_\xi^x, wJ_\xi^x - uJ_\xi^z)^T, \tag{21e}$$

where

$$D = \sqrt{(J_\xi^x)^2 + (J_\xi^y)^2 + (J_\xi^z)^2}. \tag{22}$$

Similar results hold for the Jacobian matrices of $\mathbf{G}(\mathbf{w})$ and $\mathbf{H}(\mathbf{w})$.

In the next section we develop an approximate Riemann solver based on the results of this section.

3. Approximate Riemann solver

In this section we derive an approximate Riemann solver for the solution of equations (1)–(12).

We propose solving equations (1)–(12) using operator splitting, i.e., we solve successively

$$(J\mathbf{w})_t + \mathbf{F}_\xi = \mathbf{0}, \tag{23a}$$

$$(J\mathbf{w})_t + \mathbf{G}_\eta = \mathbf{0}, \tag{23b}$$

$$(J\mathbf{w})_t + \mathbf{H}_\zeta = \mathbf{0} \tag{23c}$$

along ξ, η and ζ coordinate lines, respectively. We describe the scheme for solving equation (23a) and the solution of equations (23b) and (23c) will follow in a similar way.

3.1. Parameterisation of the equation of state

The equation of state for an ideal gas is given by

$$p = (\gamma - 1)\varrho i \tag{24}$$

where γ is a constant and represents the ratio of specific heat capacities of the fluid. Following this, for a general equation of state $p = p(\varrho, i)$ we define a new dependent variable $\gamma = \gamma(\varrho, i)$ by

$$\gamma(\varrho, i) = \frac{p(\varrho, i)}{\varrho i} + 1, \tag{25}$$

so that the equation of state (12) can be rewritten as

$$p = (\gamma(\varrho, i) - 1)\varrho i. \tag{26}$$

(Many equations of state for real gases are already given in the form of equation (26). The ideal equation of state is given by $\gamma \equiv$ constant.)

From equation (25), the eigenvectors $\mathbf{e}_{1,2}$ of equations (21a–b) can be rewritten in terms of γ as

$$\mathbf{e}_{1,2} = \left(1, u \pm \frac{aJ_\xi^x}{D}, v \pm \frac{aJ_\xi^y}{D}, w \pm \frac{aJ_\xi^z}{D}, \frac{\gamma p}{(\gamma - 1)\varrho} + \tfrac{1}{2}q^2 \pm \frac{aU}{D}\right)^T. \tag{27a–b}$$

In particular, for the ideal equation of state (24) the sound speed a is given by equation (19) as

$$a^2 = \frac{\gamma p}{\varrho}, \tag{28}$$

and the fifth component of $\mathbf{e}_3$, given by equation (21c), becomes $\frac{1}{2}q^2$ since $i - \varrho p_\varrho / p_i = 0$.

3.2. Linearised Riemann problem

If the solution of equation (23a) is sought along a ξ coordinate line given by $\eta = \eta_0, \zeta = \zeta_0$, constants, using a finite-difference method then the solution is known at a set of discrete mesh points $(\xi, \eta, \zeta, t) = (\xi_j, \eta_0, \zeta_0, t_n)$ at any time t_n. Following Godunov [4] the approximate solution $\mathbf{w}_j^n$ to $\mathbf{w}$ at $(\xi_j, \eta_0, \zeta_0, t_n)$ can be considered as a set of piecewise constants $\mathbf{w} = \mathbf{w}_j^n$ for $\xi \in (\xi_j - \frac{1}{2}\Delta\xi, \xi_j + \frac{1}{2}\Delta\xi)$ at time t_n, where $\Delta\xi = \xi_j - \xi_{j-1}$ is a constant mesh spacing. A Riemann problem is now present at each interface $\xi_{j-1/2} = \frac{1}{2}(\xi_{j-1} + \xi_j)$ separating adjacent states $\mathbf{w}_{j-1}^n, \mathbf{w}_j^n$. We consider solving the linearised Riemann problem

$$(J\mathbf{w})_t + \tilde{A}(\mathbf{w}_{j-1}^n, \mathbf{w}_j^n)\mathbf{w}_\xi = \mathbf{0} \tag{29}$$

where $\tilde{A}_{j-1/2} = \tilde{A}(\mathbf{w}_{j-1}^n, \mathbf{w}_j^n)$ is an approximation to the Jacobian matrix A and is a constant matrix depending on the states either side of $\xi_{j-1/2}$. The matrix $\tilde{A}_{j-1/2}$ will be required to satisfy the following properties

(i) $\tilde{A}_{j-1/2}$ has five linearly independent eigenvectors

and

(ii) $\Delta\mathbf{F} = \tilde{A}_{j-1/2}\Delta\mathbf{w}$.

These properties were shown by Roe [1] in the ideal-gas case in Cartesian coordinates to guarantee conservation and have good one-dimensional shock-capturing properties.

3.3. Numerical scheme

Once such a matrix has been constructed equation (29) can be solved approximately as

$$J_{j-1/2}\frac{(\mathbf{w}_k^{n+1}-\mathbf{w}_k^n)}{\Delta t}+\tilde{A}_{j-1/2}\frac{(\mathbf{w}_j^n-\mathbf{w}_{j-1}^n)}{\Delta\xi}=\mathbf{0} \tag{30}$$

where k can be $j-1$ or j, $\Delta t=t_{n+1}-t_n$ is a constant time step and $J_{j-1/2}$ is an approximation to the grid Jacobian at $(\xi,\eta,\zeta)=(\xi_{j-1/2},\eta_0,\zeta_0)$. If we project

$$\Delta\mathbf{w}=\mathbf{w}_j^n-\mathbf{w}_{j-1}^n=\sum_{i=1}^{5}\tilde{\alpha}_i\tilde{\mathbf{r}}_i \tag{31}$$

where $\tilde{\mathbf{r}}_i$ are the eigenvectors of $\tilde{A}_{j-1/2}$, then equation (30) can be written as

$$J_{j-1/2}\frac{(\mathbf{w}_k^{n+1}-\mathbf{w}_k^n)}{\Delta t}+\frac{\sum_{i=1}^{5}\tilde{\lambda}_i\tilde{\alpha}_i\tilde{\mathbf{r}}_i}{\Delta\xi}=\mathbf{0} \tag{32}$$

where $\tilde{\lambda}_i$ are the eigenvalues of $\tilde{A}_{j-1/2}$. Equation (32) now gives rise to the following first-order upwind algorithm.

$$\mathbf{w}_{j-1}^{n+1}=\mathbf{w}_{j-1}^n-\frac{\Delta t}{J_{j-1/2}\Delta\xi}\tilde{\lambda}_i\tilde{\alpha}_i\tilde{\mathbf{r}}_i \quad \text{if} \quad \tilde{\lambda}_i<0, \tag{33a}$$

or

$$\mathbf{w}_j^{n+1}=\mathbf{w}_j^n-\frac{\Delta t}{J_{j-1/2}\Delta\xi}\tilde{\lambda}_i\tilde{\alpha}_i\tilde{\mathbf{r}}_i \quad \text{if} \quad \tilde{\lambda}_i>0. \tag{33b}$$

Extensions of this first-order algorithm to second order can be made [5] and to non-uniform grids [6, 7].

3.4. Grid generation and grid Jacobian

The purpose of this paper is to present an efficient Riemann solver for use with non-Cartesian body-fitted coordinates. The mapping from physical (x, y, z) space to computational (ξ,η,ζ) space can be given analytically, or constructed numerically [8]. In the case where the mapping $x=x(\xi,\eta,\zeta)$, $y=y(\xi,\eta,\zeta)$, $z=z(\xi,\eta,\zeta)$ is known analytically we can approximate $J_{j-1/2}$ in equation (30) as

$$J_{j-1/2}=J(\xi_{j-1/2},\eta_0,\zeta_0); \tag{34}$$

alternatively, $J_{j-1/2}$ can be approximated using central differences. In addition, we will need a suitable approximation $x_\eta^{j-1/2}$ for x_η at $(\xi_{j-1/2}, \eta_0, \zeta_0)$ and in the analytic case we take

$$x_\eta^{j-1/2} = x_\eta(\xi_{j-1/2}, \eta_0, \zeta_0) \tag{35}$$

as in equation (34); otherwise we set $x_\eta^{j-1/2}$ to be the arithmetic mean of central difference approximations to x_η at $(\xi_{j-1}, \eta_0, \zeta_0)$ and (ξ_j, η_0, ζ_0). Similar approximations hold for y_η, x_ξ, x_ζ etc., and these give rise to an obvious approximation $(J_\xi^x)_{j-1/2}$ for J_ξ^x, etc.

3.5. Construction of $\tilde{A}_{j-1/2}$

Consider a ξ coordinate line given by $\eta = \eta_0$, $\zeta = \zeta_0$, constants, and denote points ξ_{j-1}, ξ_j on this line by ξ_L, ξ_R, respectively. In addition, we denote $\mathbf{w}_{j-1}^n = \mathbf{w}_L$, $\mathbf{w}_j^n = \mathbf{w}_R$, and assume that $X = (J_\xi^x)_{j-1/2}$, $Y = (J_\xi^y)_{j-1/2}$ and $Z = (J_\xi^z)_{j-1/2}$ denote approximations to J_ξ^x, J_ξ^y and J_ξ^z, respectively, that are constant in the interval (ξ_L, ξ_R). Our aim is to construct a matrix $\tilde{A}_{j-1/2} = \tilde{A}(\mathbf{w}_L, \mathbf{w}_R)$ satisfying properties (i) and (ii) of Section 3.2. Equivalently, we could find average eigenvalues $\tilde{\lambda}_i$ and average eigenvectors $\tilde{\mathbf{r}}_i$ of the eigenvalues and eigenvectors of the Jacobian matrix A at ξ_L, ξ_R given by equations (20a)–(22) such that

$$\Delta\mathbf{w} = \sum_{i=1}^{5} \tilde{\alpha}_i \tilde{\mathbf{r}}_i \tag{36a–e}$$

and

$$\Delta\mathbf{F} = \sum_{i=1}^{5} \tilde{\lambda}_i \tilde{\alpha}_i \tilde{\mathbf{r}}_i \tag{37a–e}$$

for some wavestrengths $\tilde{\alpha}_i$, where

$$\Delta(\cdot) = (\cdot)_R - (\cdot)_L. \tag{38}$$

This yields the following approximate Jacobian matrix

$$\tilde{A}_{j-1/2} = \tilde{M}_{j-1/2} \tilde{D}_{j-1/2} \tilde{M}_{j-1/2}^{-1} \tag{39}$$

with the required properties, where $\tilde{M}_{j-1/2} = [\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \tilde{\mathbf{r}}_3, \tilde{\mathbf{r}}_4, \tilde{\mathbf{r}}_5]$ and $\tilde{D}_{j-1/2} = \text{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4, \tilde{\lambda}_5)$. The choice of wavestrengths in equations (36a–36e) is made by initially considering states $\mathbf{w}_L$ and $\mathbf{w}_R$ that are close to some average state $\mathbf{w}$ as follows.

3.5. Wavespeeds for nearby states

Consider two (constant) adjacent states $\mathbf{w}_L$, $\mathbf{w}_R$ (left and right) close to an average state $\mathbf{w}$, at points L and R on a ξ-coordinate line. In particular, the variable γ given by equation (25) is piecewise constant. Now in view of the sound speed a for ideal gases ($\gamma \equiv$ constant) given by equation (28) and the eigenvectors $\mathbf{e}_{1,2}$ given by equations (27a–b), we assume that we have

approximate eigenvectors

$$\mathbf{r}_{1,2} = \left(1, u \pm \frac{aX}{d}, v \pm \frac{aY}{d}, w \pm \frac{aZ}{d}, \frac{a^2}{\gamma - 1} + \tfrac{1}{2}(u^2 + v^2 + w^2) \pm \frac{aU}{d}\right)^T \quad (40a\text{–}b)$$

where X, Y and Z are described in Section 3.5,

$$d = \sqrt{X^2 + Y^2 + Z^2} \quad (41)$$

and

$$U = Xu + Yv + Zw \quad (42)$$

corresponding to the average state $\mathbf{w}$. (N.B. The quantity γ in equations (40a–b) represents an average value close to γ_L and γ_R.) In addition, because $i - \varrho p_\varrho / p_i = 0$ for an ideal gas, we split $\mathbf{e}_3$ into two vectors as

$$\mathbf{r}_3' = (1, u, v, w, \tfrac{1}{2}(u^2 + v^2 + w^2))^T \quad (43)$$

and

$$\mathbf{r}_6' = (0, 0, 0, 0, \beta)^T, \quad (44)$$

where β represents an average value in the cell (ξ_L, ξ_R) of $i - \varrho p_\varrho / p_i$. Finally, we approximate $\mathbf{e}_{4,5}$ as

$$\mathbf{r}_4 = (0, -Y, X, 0, Xv - Yu)^T \quad (45)$$

and

$$\mathbf{r}_5 = (0, -Z, 0, X, Xw - Zu)^T. \quad (46)$$

We seek coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ such that

$$\Delta\mathbf{w} = \alpha_1\mathbf{r}_1 + \alpha_2\mathbf{r}_2 + \alpha_3\mathbf{r}_3' + \alpha_4\mathbf{r}_4 + \alpha_5\mathbf{r}_5 + \mathbf{r}_6' \quad (47)$$

to within $O(\Delta^2)$. (N.B. The vector $\mathbf{r}_6'$ is considered separately since it vanishes for an ideal gas. Also, we do not introduce another coefficient α_6 since $\mathbf{r}_6'$ has only one non-zero component and is therefore not required.) After some manipulation we find that equation (47) yields the following expressions for α_i and β

$$\alpha_{1,2} = \frac{1}{2a^2}\left(\Delta p \pm \frac{\varrho a \Delta U}{d}\right)^T, \quad (48a\text{–}b)$$

$$\alpha_3 = \Delta\varrho - \frac{\Delta p}{a^2}, \quad (48c)$$

$$\alpha_4 = \frac{\varrho \Delta v}{X} - \frac{\varrho Y \Delta U}{d^2 X}, \tag{48d}$$

$$\alpha_5 = \frac{\varrho \Delta w}{X} - \frac{\varrho Z \Delta U}{d^2 X} \tag{48e}$$

and

$$\beta = -\frac{\varrho i \Delta \gamma}{\gamma - 1}, \tag{49}$$

where we have made the assumption that to within $O(\Delta^2)$

$$\Delta(\varrho N) = N \Delta \varrho + \varrho \Delta N, \qquad N = u, v \text{ or } w, \tag{50a–c}$$

$$\Delta(\varrho N^2) = N^2 \Delta \varrho + 2 \varrho N \Delta N, \quad N = u, v \text{ or } w, \tag{51a–c}$$

and

$$\Delta\left(\frac{p}{\gamma - 1}\right) = \frac{\Delta p}{\gamma - 1} - \frac{p}{(\gamma - 1)^2} \Delta \gamma. \tag{52}$$

The results above imply an approximation to the eigenvector $\mathbf{e}_3$ given by $\mathbf{r}_3 = \mathbf{r}_3' + \mathbf{r}_6'/\alpha_3$, i.e.

$$\mathbf{r}_3 = \left(1, u, v, w, \tfrac{1}{2}(u^2 + v^2 + w^2) - \frac{\varrho i \Delta \gamma}{(\gamma - 1)\left(\Delta \varrho - \dfrac{\Delta p}{a^2}\right)}\right)^T, \tag{53}$$

and hence an approximation to $i - \varrho p_\varrho / p_i$. With the expressions given above it is possible to show that

$$\Delta \mathbf{F} = \sum_{i=1}^{5} \lambda_i \alpha_i \mathbf{r}_i \tag{54}$$

to within $O(\Delta^2)$. We now return to the general case.

3.7. Decomposition for general $\mathbf{w}_L$, $\mathbf{w}_R$

Consider two states $\mathbf{w}_L$, $\mathbf{w}_R$ not necessarily close such that equations (36a)–(37e) are satisfied exactly, where

$$\tilde{\lambda}_i = \tilde{U} \pm \tilde{a} d, \tilde{U}, \tilde{U}, \tilde{U}, \tag{55a–e}$$

$$\tilde{\mathbf{r}}_{1,2} = \left(1, \tilde{u} \pm \frac{\tilde{a} X}{d}, \tilde{v} \pm \frac{\tilde{a} Y}{d}, \tilde{w} \pm \frac{\tilde{a} Z}{d}, \frac{\tilde{a}^2}{\tilde{\gamma} - 1} + \tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\tilde{v}^2 + \tfrac{1}{2}\tilde{w}^2 \pm \frac{\tilde{a}\tilde{U}}{d}\right)^T, \tag{56a–b}$$

$$\tilde{\mathbf{r}}_3 = \left(1, \tilde{u}, \tilde{v}, \tilde{w}, \tfrac{1}{2}(\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) - \frac{\tilde{\varrho}\tilde{i}\Delta\gamma}{(\tilde{\gamma} - 1)\left(\Delta\varrho - \frac{\Delta p}{\tilde{a}^2}\right)}\right)^T, \tag{56c}$$

$$\tilde{\mathbf{r}}_4 = (0, -Y, X, 0, X\tilde{v} - Y\tilde{u}), \tag{56d}$$

$$\tilde{\mathbf{r}}_5 = (0 -Z, 0, X, X\tilde{w}, -Z\tilde{u})^T, \tag{56e}$$

$$\tilde{\alpha}_{1,2} = \frac{1}{2\tilde{a}^2}\left(\Delta p \pm \frac{\tilde{\varrho}\tilde{a}\Delta U}{d}\right), \tag{57a–b}$$

$$\tilde{\alpha}_3 = \Delta\varrho - \frac{\Delta p}{\tilde{a}^2}, \tag{57c}$$

$$\tilde{\alpha}_4 = \frac{\tilde{\varrho}\Delta v}{X} - \frac{\tilde{\varrho}Y\Delta U}{d^2 X}, \tag{57d}$$

$$\tilde{\alpha}_5 = \frac{\tilde{\varrho}\Delta w}{X} - \frac{\tilde{\varrho}Z\Delta U}{d^2 X}, \tag{57e}$$

$$\tilde{U} = X\tilde{u} + Y\tilde{v} + Z\tilde{w}, \tag{58}$$

$$\gamma = \frac{p(\varrho, i)}{\varrho i} + 1 \tag{59}$$

and

$$\Delta U = \Delta(Xu + Yv + Zw) = X\Delta u + Y\Delta v + Z\Delta w. \tag{60}$$

(N.B. X, Y and Z are constant in (ξ_L, ξ_R).) Thus, we have to determine averages $\tilde{\varrho}$, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{a}$, $\tilde{i}$, and $\tilde{\gamma}$ such that equations (36a)–(37e) are satisfied subject to equations (55a)–(60). The solution to this problem can be determined and gives the following averages

$$\tilde{N} = \frac{\sqrt{\varrho_L}N_L + \sqrt{\varrho_R}N_R}{\sqrt{\varrho_L} + \sqrt{\varrho_R}}, \quad N = u, v, w, i, \gamma \text{ or } H, \tag{61a–f}$$

$$\tilde{\varrho} = \sqrt{\varrho_L \varrho_R}, \tag{62}$$

$$\tilde{a}^2 = (\tilde{\gamma} - 1)[\tilde{H} - \tfrac{1}{2}(\tilde{u}^2 + v^2 + \tilde{w}^2)]. \tag{63}$$

where

$$H = \frac{p}{\varrho} + i + \tfrac{1}{2}(u^2 + v^2 + w^2) \tag{64}$$

$$= \frac{\gamma p}{\varrho(\gamma - 1)} + \tfrac{1}{2}(u^2 + v^2 + w^2)$$

is the enthalpy.

Similar results hold for updating in the η and ζ directions.

In the next section we give the numerical results for a test problem using the algorithm of this section.

4. Numerical results

In this section we give the numerical results for a standard test problem in two-dimensional gas dynamics using the Riemann solver described in Section 3.

The problem is that of uniform flow of "real air" past a circular cylinder. The equation of state used can be written as

$$p = (\gamma(\varrho, i) - 1)\varrho i$$

where the form of $\gamma(\varrho, i)$ is determined via curve fits to experimental data [3]. The radius of the cylinder is 0.5 and the initial conditions are $\varrho = 1.4, u = 8.0, v = w = 0.0, p = 1.0$ which corresponds to Mach 8 flow. An O-type computational mesh is used and thus the grid transformation is from (x, y, z) physical space to $(\xi, \eta, \zeta) \equiv (R, \phi, z)$ computational space, where R, ϕ, z are standard cylindrical polar coordinates. The region of computation considered is $(R, \phi) \in [0.5, 20] \times [0, 2\pi]$ and we apply periodic conditions along $\phi = 0$.

The grid spacing in the ϕ-direction is uniform with 128 grid lines given by $\phi_j = (j - \frac{1}{2})\pi/128$, $j = 1, \ldots, 128$. In the R-direction the grid spacing is geometric with 33 grid lines given by $R_1 = 0.5 + \frac{1}{2}k$, $r_j = R_{j-1} + k\mu^{j-2}$, $j = 2, \ldots, 33$, where $k = \pi/128$ and $\mu = 1.1648336$. Figure 1 displays the isomach contours for Mach 8 flow after 1000 time steps

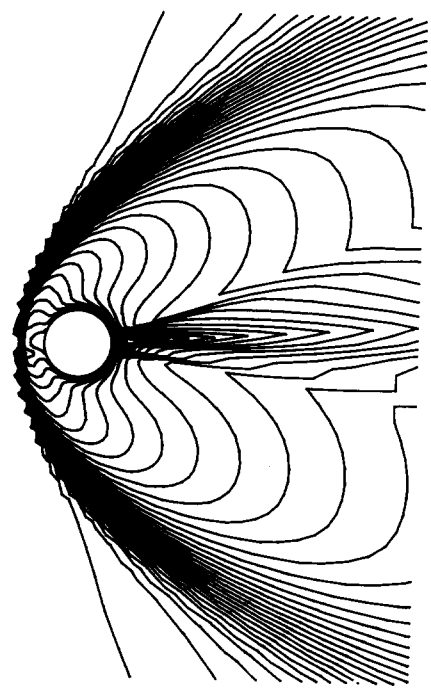

Fig. 1. Isomach contours for Mach 8 flow in $0.5 . \leqslant R \leqslant 5.0$. There are 31 contours corresponding to an equal spacing of the interval from Mach 0 to Mach 8.

in the same region. We note, however, the slight asymmetry of Figure 1 which is a result of using an operator split scheme, i.e. one which is not genuinely two-dimensional.

In all cases the shock has been captured over at most three cells.

5. Conclusions

We have presented an efficient Riemann solver for three-dimensional compressible flows using body-fitted coordinates. The scheme applies to a general convex equation of state and by using a local parameterisation of the equation of state, only one function call is required per one-dimensional computational cell. The numerical results achieved show that the shock has been captured over two or three cells.

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